Basic Concepts: Sets, Relations and Mappings

1.1 Introduction

In this initial chapter we recall the concepts of *sets* and *functions* which are fundamental in the study of real analysis.

In every language there are certain terms which are basic and remain undefined but whose meanings are universally accepted. In Mathematics the word *set* is such a term: a set is understood to be a well-defined collection of distinct objects called *elements*. The term well-defined is that properly of the set by which one is able to determine whether a given element belongs to the set, or not.

Some authors prefer to take the word *set* as a primary and an undefined concept and then develop it axiomatically (as in the book *Axiomatic Set Theory* by P. Suppes).

Our understanding of *well-defined collection of distinct objects* is intuitive and naive and adequate for our purpose.

About Sets. We shall identify a set by stating its *members* (or *elements*). We denote sets by capital letters A, B, C, etc. and use lower case letters a, b, c, etc. to represent their elements.

If an element x is in the set A, we write $x \in A$ and say that x is a *member* of A or *x belongs to A*. If x is not in A, we write $x \notin A$ (x does not belong to A).

We write $\{x\}$ to denote a *singleton set* whose only member is x.

We write $\{x_1, x_2, \cdots, x_n\}$ to denote a finite set of n elements $x_1, x_2, \cdots, x_n$.

We may write an infinite set like $\mathbb{N} = \{1, 2, 3, \cdots\}$, the set of all natural numbers, where we use a curly bracket to enclose some elements and three dots to imply the existence of other elements.

When it is possible to list all the elements of a finite set, we call it *Roaster Method of representation* of sets. e.g., $\{a, e, i, o, u\}$, the set of five vowels in the English alphabet. But most often a set is represented by some specific property $P(x)$ common to all elements of the set. We write

$$X = \{x : x \text{ obeys } P(x)\} \text{ or } simply\ X = \{x : P(x)\}.$$

We shall, throughout this text, use the following notations for some specific sets:

$\mathbb{N}$ = set of all natural numbers or positive integers = $\{1, 2, 3, \cdots\}$.

$\mathbb{Z}$ = set of all integers = $\{0, \pm 1, \pm 2, \cdots\}$.

$\mathbb{Q}$ = set of all rational numbers = $\{p/q : p \in \mathbb{Z} \text{ and } q \in \mathbb{N}\}$

$\mathbb{R}$ = set of all real numbers.

We write $\mathbb{Z}^+$, $\mathbb{Q}^+$, $\mathbb{R}^+$ to denote the positive elements in the respective sets.

Subsets. If each element in the set A is also a member of the set B, then we say that A is a *subset* of B and we write $A \subseteq B$ (A is included in B) or equivalently, $B \supseteq A$ (B includes A).

We say that a set A is a *proper subset* of B if $A \subseteq B$ and $\exists$ at least one element of B that is not in A.

We then write: $A \subset B$ (read: A is a proper subset of B).

Equality of two sets. Two sets A and B are said to be equal if the sets consist of precisely the same elements. We then write $A = B$.

It is easy to see that $A = B$, provided $A \subseteq B$ and $B \subseteq A$ and conversely,

$$A \subseteq B \text{ and } B \subseteq A \Longrightarrow A = B.$$

Universal set U and Empty set ϕ. In any discussion involving sets we consider a fixed set U which is the set of all elements under discussion. Thus every set in that discussion is a subset of U. We call U the *universal set* or *the universe.*

In real analysis the set $\mathbb{R}$ of all real numbers is taken as the universe. We deal, therefore, with subsets of real numbers.

Note: We also use S for an universal set.

The symbol ϕ denotes what we call the *empty set* or *null set* which contains no element. For e.g., the set whose elements are common elements of $\{2, 3, 4\}$ and $\{5, 8, 7\}$ is the null set ϕ. The set of all students of Class V of a school who secured more than 90 out of 100 is ϕ, if the highest mark is 90. We do not know in advance whether any student secured more than 90 or not. That is why, such a conceptual set is necessary in theory of sets. For logical consistency ϕ is a subset of any set A. It is called an improper subset of any set A.

Remember: For every set A we have $\phi \subseteq A \subseteq U$ and $A \subseteq A$. [A is called trivial subset of A]

An useful result. *Set inclusion is transitive,* i.e., if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

[For, $A \subseteq B \Longrightarrow x \in A \Longrightarrow x \in B$; $B \subseteq C \Longrightarrow x \in B \Longrightarrow x \in C$.

$\therefore$ two together $\Longrightarrow x \in A \Longrightarrow x \in C$, i.e., $A \subseteq C$]

Family or Collection of Sets. Let I be any set. Suppose for each member $i \in I$ we can associate a set A_i. Then the collection $\{A_i : i \in I\}$ form a *family of sets indexed by I* (I is known as an *index set*).

Power set. Given a set A. We collect the family of all subsets of A (this family includes the set A itself and ϕ). This family of sets is called the *power set* of A, denoted by $P(A)$.

As for example, let $A = \{1, 2, 3\}$. Then

$$P(A) = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, \phi\}.$$

In fact, if A is a finite set of n elements, then $P(A)$ has 2^n elements.

[*Hints:* ${}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n$]

1.2 Operations on Sets (Set Algebra)

We now define methods of obtaining new sets from given ones—these methods are called *operations on sets.* Some of those operations—*Union, Intersection, Complementation* and *Difference of two sets,* are described below:

I. *Union*: The *union* of two sets A and B is the set $A \cup B = \{x : x \in A \text{ or } x \in B\}$.

(The word *or* is to be used in the *inclusive sense* allowing the possibility that x may belong to both the sets).

For the collection of sets A_i indexed by $i \in I$ we define the union of this collection by $\bigcup_{i \in I} A_i = \{x : x \in A_i \textit{ for some } i \in I\}$, where I is an index set.

[In case $I = \mathbb{N}$ = set of all positive integers n, the union is denoted by

$\bigcup_{n=1}^{\infty} A_n = \{x : x \in A_n \text{ for some } n \in \mathbb{N}\}$; it has a special name—*countable union*]

II. *Intersection*: The intersection of two sets A and B is the set

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

For the collection of sets A_i indexed by $i \in I$, the intersection

$$\bigcap_{i \in I} A_i = \{x : x \in A_i \text{ for all } i \in I\}$$

and the countable intersection (i.e., when $I = \mathbb{N}$)

$$\bigcap_{i=1}^{\infty} A_i = \{x : x \in A_i \text{ for all } i \in \mathbb{N}\}.$$

Examples (i) Let $A_n = \{n\}$. Then $\bigcup_{n=1}^{\infty} A_n = \mathbb{N}$ and $\bigcup_{n=-\infty}^{\infty} A_n = \mathbb{Z}$, $\bigcap_{n=1}^{\infty} A_n = \phi$.

(ii) Let $A_n = (-1/n, 1/n)$, $n \in \mathbb{N}$. Then $\bigcup_{n=1}^{\infty} A_n = (-1, 1)$ and $\bigcap_{n=1}^{\infty} A_n = \{0\}$.

(iii) Let $A_n = \{1, 2, 3, \cdots, n\}$. Then $\bigcup_{n \in \mathbb{N}} A_n = \mathbb{N}$ but $\bigcap_{n \in \mathbb{N}} A_n = \{1\}$.

Disjoint sets. Two sets A and B are said to be disjoint, if they have no elements in common, i.e., if $A \cap B = \phi$.

The family of sets is called *pairwise disjoint,* if each distinct pair of elements of the collection are disjoint. Thus an *indexed collection* $\{A_i\}_{i \in I}$ is pairwise disjoint, if $A_i \cap A_j = \phi$ for all $i, j \in I$ and $i \neq j$.

III. *Complementation.* Let U be the universal set. Suppose that A and B are two subsets of U. Then we define *the complement (or difference) of B relative to A,* denoted by $A - B$ or $A \backslash B$ (A slash B), to be the set

$$A - B = \{x : x \in U, x \in A \text{ and } x \notin B\}.$$

By A' or A^c (complement of A) we mean $U - A$, i.e.,

$A^c = \{x : x \in U,\ x \notin A\} = \{x : x \notin A\}$ ($\because x$ always belongs to U, no need to mention).

In *Real Analysis,* the universe is $\mathbb{R}$, the set of all real numbers. If $A \subseteq \mathbb{R}$, then

$$A' \text{ or } A^c = \mathbb{R} - A = \mathbb{R} \backslash A = \{x : x \in \mathbb{R} \text{ and } x \notin A\} = \{x : x \notin A\}.$$

Example (i) Let $A = \{2, 4, 6\}$ and $B = \{2, 6, 10, 14\}$. Then complement of B relative to A is the set

$A \backslash B = A - B = \{x : x \in A \text{ and } x \notin B\} = \{4\}$ and

$B \backslash A = B - A = \{x : x \in B \text{ and } x \notin A\} = \{10, 14\}$.

Note that $A - B$ and $B - A$ are *two disjoint sets.*

(ii) Let $\mathbb{Q} \subset \mathbb{R}$ ($\mathbb{Q}$ is the set of all rational numbers).

Then $\mathbb{Q}' = \mathbb{R} - \mathbb{Q} = \{$set of all irrational numbers$\}$.

IMPORTANT CONSEQUENCES

Let U be the universal set and $A, B, C \subseteq U$. Then

1. $A \cup B = B \cup A$ and $A \cap B = B \cap A$. Union and Intersection are commutative.
2. $A \cup (B \cup C) = (A \cup B) \cup C$; $A \cap (B \cap C) = (A \cap B) \cap C$. Union and intersection are associative.
3. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$; $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Union is distributive over intersection and intersection is distributive over union, i.e., each is distributive over the other.
4. De Morgan's laws: $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$.
5. $A - B = A \cap B'$.

Besides Union, Intersection and Complementation we introduce *two important operations:*

IV. *Symmetric difference:* Symmetric difference of two sets A and B is denoted by $A \triangle B$ and is defined by

$$A \triangle B = (A - B) \cup (B - A).$$

V. *Cartesian product:* If A and B are two non-empty sets then the *Cartesian product* $A \times B$ of A and B is the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$, that is, $A \times B = \{(a, b) : a \in A, b \in B\}$.

For e.g., let $A = \{1, 2, 3\}$ and $B = \{1, 4\}$, then the Cartesian product $A \times B$ is the set whose members are $(1, 1)$, $(1, 4)$, $(2, 1)$, $(2, 4)$, $(3, 1)$, $(3, 4)$.

We may visualize that the set $A \times B$ corresponds to six points on the plane with coordinates that we have listed above.

We may draw a diagram (Fig. 1.2.1) to exhibit the elements of $A \times B$.

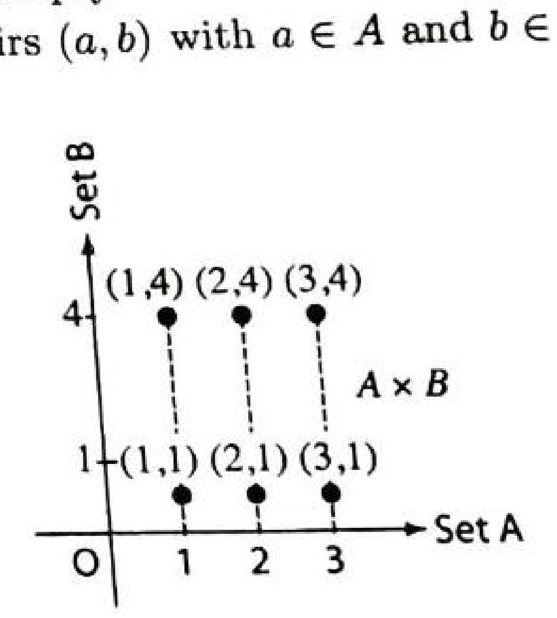

Fig 1.2.1

It is interesting to draw the diagram of $A \times B$, if

$$A = \{x : x \in \mathbb{R} \text{ and } 1 \leq x \leq 2\}$$

and $B = \{y : y \in \mathbb{R} \text{ and } 0 \leq y \leq 1 \text{ or } 2 \leq y \leq 3\}$.

The diagram of $A \times B$ is the adjoined.

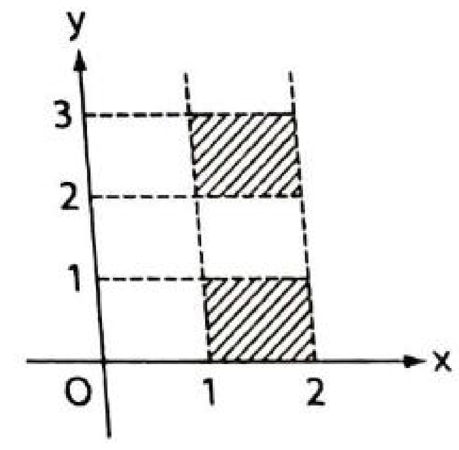

Fig 1.2.2

Properties: Symmetric difference and Cartesian products

1. Symmetric difference is both commutative and associative:
   $A \triangle B = B \triangle A$;
   $A \triangle (B \triangle C) = (A \triangle B) \triangle C$.
2. We have defined $A \triangle B = (A-B) \cup (B-A)$. From this we can show that $A \triangle B = (A \cup B) - (A \cap B)$ (see Ex. 1.3.5). Draw a diagram of $A \triangle B$.
3. $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$.
4. $A \triangle \phi = A$ and $A \triangle A = \phi$.
5. *Cartesian products*:
   (a) $A \times B \neq B \times A$; $A \times B = B \times A \iff A = B$;
   (b) $(A \cup B) \times C = (A \times C) \cup (B \times C)$; $A \times (B \cup C) = (A \times B) \cup (A \times C)$;
   (c) $(A \cap B) \times C = (A \times C) \cap (B \times C)$; $A \times (B \cap C) = (A \times B) \cap (A \times C)$;
   (d) $(A - B) \times C = (A \times C) - (B \times C)$; $A \times (B - C) = (A \times B) - (A \times C)$;
   (e) $A \neq \phi$, $A \times B = A \times C \implies B = C$.
6. *Extension*: Let $A_1, A_2, \cdots, A_n$ be a finite collection of n sets. Then
   $A_1 \times A_2 \times A_3 \times \cdots \times A_n = \{(a_1, a_2, \cdots, a_n) : a_1 \in A_1, a_2 \in A_2 \cdots a_n \in A_n\}$.
   In case $A_1 = A_2 = \cdots = A_n = A$ (say), the Cartesian product
   $$A^n = \{(a_1, a_2, \cdots, a_n) : a_i \in A\}.$$
   Let $A = \mathbb{R}$. Then $\mathbb{R}^n = \{(a_1, a_2, \cdots, a_n); a_i \in \mathbb{R}\}$.
   We call $(a_1, a_2, \cdots, a_n)$, where each $a_i \in \mathbb{R}$, an *n-tuple* of real numbers.

Venn diagram. For the purpose of illustrations we may often use *Venn diagrams*. Given below Venn diagram representations of the different operations on sets:

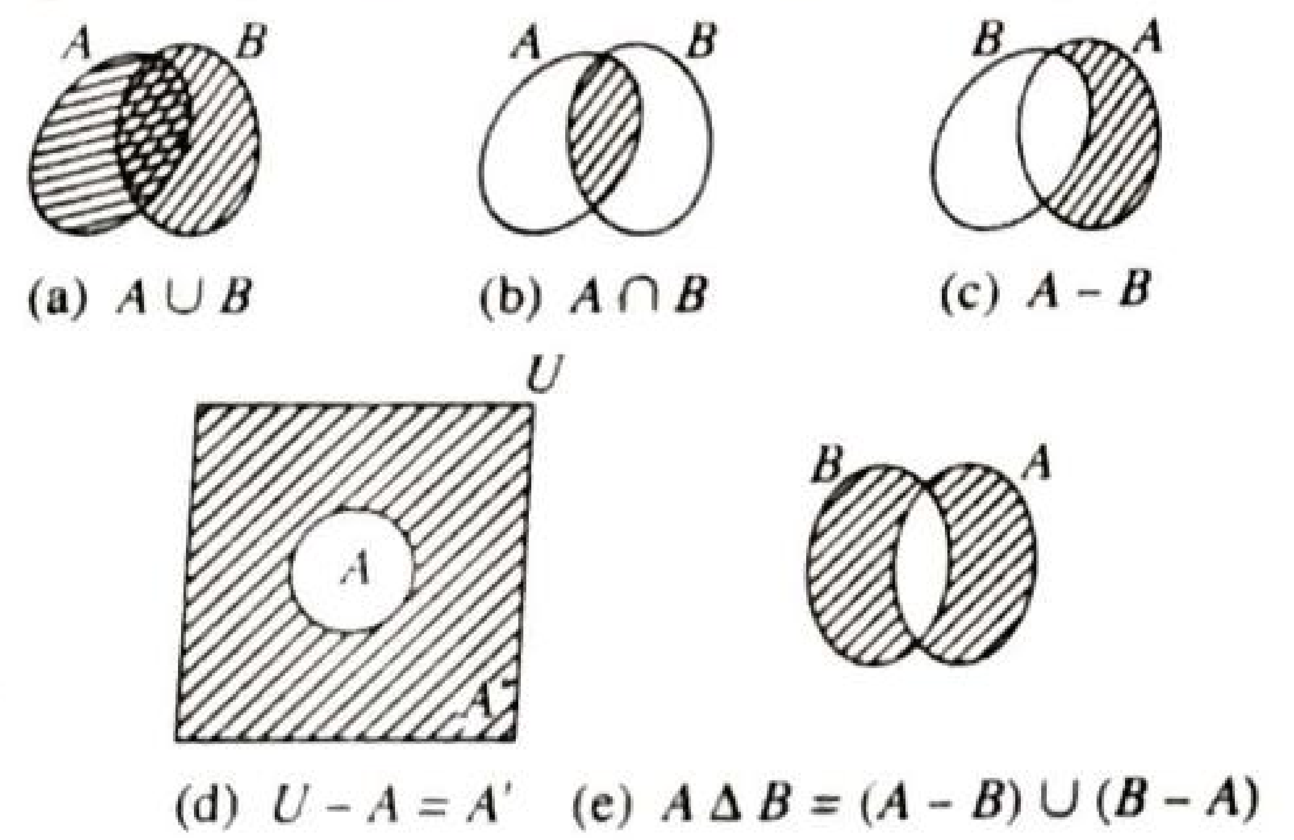

Fig 1.2.3 Diagrammatic representation of operations of sets.

1.3 Solved Examples

Example 1.3.1. *Let* $A = \{1, 1/2, 1/3, 1/4, 1/5\}$ *and* $B\{1/2, 1/5, 1/8, 1/100\}$. *Determine the elements of the set* $A \cup (A \cap B)$.

Solution: $A \cap B = \{1/2, 1/5\}$ and hence $A \cup (A \cap B)$,
i.e., $\{1, 1/2, 1/3, 1/4, 1/5\} \cup \{1/2, 1/5\} = \{1, 1/2, 1/3, 1/4, 1/5\}$.

Example 1.3.2. *If the universe* $S = \{1, 2, 3, 4, 5, 6\}$ *and if* A, B, C *are three subsets of* S, *where* $A = \{1, 3, 4, 6\}$ *and* $B \cap C = \{1, 2, 6\}$, *then determine the set:*
(i) $(A \cup B) \cap (A \cup C)$; (ii) $B' \cup C'$.

Solution: (i) $(A \cup B) \cap (A \cup C) = A \cup (B \cap C)$ (using distributive law)
$= \{1, 3, 4, 6\} \cup \{1, 2, 6\}$
$= \{1, 2, 3, 4, 6\}$.

(ii) $B' \cup C' = (B \cap C)' = S - (B \cap C) = \{1, 2, 3, 4, 5, 6\} - \{1, 2, 6\} = \{3, 4, 5\}$.

Note: In solving problems on set equalities, e.g., to prove $X = Y$, we show $X \subseteq Y$ and $Y \subseteq X$. Another useful relation is $A - B = A \cap B'$.

Example 1.3.3. *If A, B, C are three sets, then* (i) $A - (B \cup C) = (A - B) \cap (A - C)$; (ii) $A - (B \cap C) = (A - B) \cup (A - C)$.

Solution: To prove (i), we shall prove that every element of LHS $A - (B \cup C)$ is contained in both $(A - B)$ and $(A - C)$ and conversely.

Let $x \in A - (B \cup C)$, then $x \in A$ and $x \notin B \cup C$. Hence, $x \in A$ and x is neither in B nor in C. Therefore, $x \in A$ but $x \notin B$ and $x \in A$ but $x \notin C$.

Then $x \in A - B$ as well as $x \in A - C$. $\therefore$ $x \in (A - B) \cap (A - C)$.

$$\therefore\ A - (B \cup C) \subseteq (A - B) \cap (A - C). \quad (1)$$

Conversely, if $x \in (A - B) \cap (A - C)$, then $x \in A - B$ and $x \in A - C$. Hence, $x \in A$ and $x \notin B$ and $x \notin C$. Therefore, $x \in A$ and $x \notin (B \cup C)$, i.e., $x \in A - (B \cup C)$.

$$\therefore\ (A - B) \cap (A - C) \subseteq A - (B \cup C). \quad (2)$$

Relations (1) and (2)

$$A - (B \cap C) = (A - B) \cap (A - C)$$

To prove (ii), proceed exactly in a similar manner.

Example 1.3.4. *Let S be the universal set.* A, B, C *are any three subsets of S. Then*

(i) $A \cap (B - C) = (A \cap B) - (A \cap C)$;
(ii) $(A' \cup B') \cup (A \cap B \cap C') = A' \cup B' \cup C'$;
(iii) $(A' \cap B' \cap C) \cup (B \cap C) \cup (A \cap C) = C$;

where A', B', C' are respectively the complements of A, B, C relative to S.

Solution: (i) RHS $= (A \cap B) - (A \cap C) = (A \cap B) \cap (A \cap C)'$
$= (A \cap B) \cap (A' \cup C')$ (De Morgan's law)
$= \{(A \cap B) \cap A'\} \cup \{(A \cap B) \cap C'\}$ (Distribution law)
$= \phi \cup \{(A \cap B) \cap C'\} = (A \cap B) \cap C'$

LHS $= A \cap (B - C) = A \cap (B \cap C') = (A \cap B) \cap C'$ (Associative law)

$\therefore\ A \cap (B - C) = (A \cap B) - (A \cap C)$, proved.

Note: $(A \cap B) \cap A' = A' \cap (A \cap B)$ (Commutative property)
$= (A' \cap A) \cap B$ (Associative law)
$= \phi \cap B = \phi$.

(ii) LHS $= (A' \cup B') \cup (A \cap B \cap C')$
$= (A \cap B)' \cup \{(A \cap B) \cap C'\}$ (De Morgan's law)
$= \{(A \cap B)' \cup (A \cap B)\} \cap \{(A \cap B)' \cup C'\}$ (Distributive law)
$= S \cap \{A \cap B)' \cup C'\} = (A \cap B)' \cup C'$
$= (A' \cup B') \cup C'$ (De Morgan's law)
$= A' \cup B' \cup C'$ (Associative property)
$=$ RHS (proved).

(iii) First, we observe that
$(B \cap C) \cup (A \cap C) = (C \cap B) \cup (C \cap A)$ (Commutative law)
$= C \cap (B \cup A)$ (Distributive law)
$= (B \cup A) \cap C$ (Commutative law)
$= (A \cup B) \cap C$. (Commutative law)

We also have $A' \cap B' \cap C = (A' \cap B') \cap C$ (Associative law)
$= (A \cup B)' \cap C$. (De Morgan's law)

Now, the given LHS $= (A' \cap B' \cap C) \cup (B \cap C) \cup (A \cap C)$
$= \{(A \cap B)' \cap C\} \cup \{(A \cup B) \cap C\}$
$= \{(A \cup B)' \cup (A \cup B)\} \cap C$ (Distributive law)
$= S \cap C = C =$ RHS (proved).

Example 1.3.5. *Prove the following for any three subsets A, B, C of the universal set S:*

(i) $(A - C) \cap (B - C) = (A \cap B) - C$;
(ii) $(A - B) \cup B = A$ *iff* $B \subseteq A$.
(iii) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$, i.e., $A \triangle B = (A \cup B) - (A \cap B)$.

(iv) *If $A \triangle B = A \triangle C$, prove that $B = C$.*

Solution: (i) $A - C = A \cap C'$ and $B - C = B \cap C'$.
LHS $= (A - C) \cap (B - C)$
$= (A \cap C') \cap (B \cap C')$
$= (A \cap C') \cap (C' \cap B)$ (Commutative law for $\cap$)
$= \{(A \cap C') \cap C'\} \cap B$ (Associative law)
$= \{A \cap (C' \cap C')\} \cap B$ (Associative law)
$= (A \cap C') \cap B$ ($\because C' \cap C' = C'$)
$= B \cap (A \cap C')$ (Commutative law for $\cap$)
$= (B \cap A) \cap C'$ (Associative law)
$= (A \cap B) \cap C'$ (Commutative law for $\cap$)
$= (A \cap B) - C$
$=$ RHS (proved).

(ii) $(A - B) \cup B = (A \cap B') \cup B$
$= (A \cup B) \cap (B' \cup B)$ (Distributive law)
$= (A \cup B) \cap S$ (S = universe = $B' \cup B$)
$= A \cup B$ [$\because A \cup B \subseteq S$].

Now if $B \subseteq A$, $A \cup B = A$ and conversely $A \cup B = A \Longrightarrow B \subseteq A$.
$\therefore (A - B) \cup B = A$ *iff* $B \subseteq A$.

(iii) $(A - B) \cup (B - A) = (A \cap B') \cup (B \cap A')$
$= \{(A \cap B') \cup B\} \cap \{(A \cap B') \cup A'\}$ (Distributive law)
$= \{(A \cup B) \cap (B' \cup B)\} \cap \{(A \cup A') \cap (B' \cup A')\}$ (Distributive law)
$= \{(A \cup B) \cap S\} \cap \{S \cap (B' \cup A')$ (S = universe)
$= (A \cup B) \cap (B' \cup A')$
$= (A \cup B) \cap (A \cap B)'$ (De Morgan's law)
$= (A \cup B) - (A \cap B)$.

Thus $A \triangle B = (A \cup B) - (A \cap B)$.

(iv) $B = \phi \triangle B = (A \triangle A) \triangle B = A \triangle (A \triangle B) = A \triangle (A \triangle C) \triangle C = \phi \triangle C = C$.

1.4 Relation from A to B: Relation on a set A

Let A and B be two non-empty sets. A relation $\Re$ from A to B is any subset of the Cartesian product $A \times B$.

We often speak of a relation $\Re$ from A to A: we call it *relation on the set A*.

Definition. Let A be a non-empty set. A subset $\Re$ of the Cartesian product $A \times A$ is called a *relation on* A. If $(x, y) \in \Re$, we say that x is related to y by the relation $\Re$ and we write $x\Re y$.

e.g., $\Re = \{(1,2), (1,3), (2,3)\}$ is a relation on the set $A = \{1,2,3\}$. Here $1\Re 2$, $1\Re 3$, $2\Re 3$. Actually it is the usual relation $<$ (less than) because $1 < 2$, $1 < 3$, $2 < 3$. On the same set $A = \{1,2,3\}$, relation $\leq$ is described by the set $\{(1,1), (2,2), (3,3), (1,2), (1,3), (2,3)\}$.

We note that $<, =, >, \leq, \geq$ are relations on the sets of numbers $\mathbb{R}, \mathbb{N}, \mathbb{Z}, \mathbb{Q}$. *Is the mother of, Is the brother of, Is married to* are relation on the set of all human beings.

Definition. Let $\Re$ be a relation on a set A. For any three elements $x, yz \in A$

(i) If $x\Re x$ (i.e., $(x,x) \in \Re$) then $\Re$ is called a *reflexive relation;*

(ii) If $x\Re y \Longrightarrow y\Re x$ (i.e., whenever $(x,y) \in \Re$, (y,x) also belongs to $\Re$), then $\Re$ is said to be a *symmetric relation*;

(iii) If $x\Re y$ and $y\Re z \Longrightarrow x\Re z$ (i.e., if $(x,y) \in \Re$ and $(y,z) \in \Re$, then $(x,z) \in \Re$), then $\Re$ is said to be a *transitive relation.*

A relation $\Re$ on a set A is called an *equivalence relation on A* if $\Re$ is *reflexive, symmetric* and *transitive.*

Note: Sometimes such relations are called *binary relations.*

Moreover, $\Re$ is called *antisymmetric*, if $a\Re b$ and $b\Re a$ together imply $a = b$.

We consider a few examples on relations:

Example 1.4.1. *Let $A = \mathbb{Z}$, the set of all integers. Consider the subset $\Re$ of $\mathbb{Z} \times \mathbb{Z}$ defined by $\Re = \{(x,y) : x - y$ is divisible by $3\}$.*

Solution: Here $x\Re y$, if $(x - y)$ is divisible by 3.

(a) $\Re$ is *reflexive* ($\because x - x$ is divisible by 3 for every $x \in \mathbb{Z}$);

(b) $\Re$ is *symmetric* (because $x - y$ is divisible by $3 \Longrightarrow y - x$ is divisible by 3, i.e., $x\Re y \Longrightarrow y\Re x$).

(c) If $x - y$ is divisible by 3 and $y - z$ is divisible by 3, then it is certainly true that $x - z = (x - y) + (y - z)$ is also divisible by 3.

Thus $x\Re y$ and $y\Re z \Longrightarrow x\Re z$. Hence $\Re$ is *transitive.*

Thus here $\Re$ is an equivalence relation.

Example 1.4.2. *Let $A = \mathbb{R}$ (the set of all real numbers). It is easy to prove that the relation '=' is an equivalence relation (exactly as in the previous problem).*

Example 1.4.3. *Let $A = \mathbb{Z} - \{1\}$. We define the relation $\Re$ on this set by the rule $x\Re y$, iff x and y have common factor other than $1\}$. Verify that this relation is reflexive*

and symmetric but not transitive (R and S but not T). For e.g., $x = 12, y = 15, z = 25$, $x\Re y, y\Re x, x\Re z$.

Example 1.4.4. *Let $A = \mathbb{Z}$ = set of all integers. On $\mathbb{Z}$, we define the relation $\Re$ to mean $x > y$. This relation $\Re$ is neither reflexive nor symmetric but it is transitive. (T but not R or S)*

Example 1.4.5. *On $\mathbb{Z}$ we define $x\Re y$ to mean $x \leq y$. This relation is reflexive and transitive but not symmetric. (R and T but not S). For e.g., $3 \leq 7$ but $y \leq 3$ is not true.*

Example 1.4.6. *On $\mathbb{Z}$ we define $x\Re y$ to mean $x \leq y + 1$. This relation is reflexive, but neither symmetric nor transitive. (R but not S and T)*

Example 1.4.7. *On $\mathbb{Z}$ we define $x\Re y$ to mean $x = -y$. This relation is neither reflexive, nor transitive but it is symmetric. (S but not R and T)*

Example 1.4.8. *On the set A of all fractions of the form a/b, where a, b are integers with $a, b \neq 0$, we define $a/b \ \Re \ c/d$, iff $b = c$. This relation is neither reflexive, nor symmetric, nor transitive. (Not R, S, T)*

Example 1.4.9. *Let $A = \{1, 2, 4, 6, \cdots\}$. We define the relation $\Re$ by $x\Re y$, iff x and y have a common factor other than 1. This relation is symmetric and transitive, but it is not reflexive because $1\Re 1$ is not true. (S and T but not R)*

Example 1.4.10. *Let A be the set of all complex numbers. We define the relation $\Re$ on this set by $z\Re w$ (where z and w are two complex numbers) to mean $\mathrm{Re}(z) \leq \mathrm{Re}(w)$ and $\mathrm{Im}(z) \leq \mathrm{Im}(w)$. Then this relation is reflexive and transitive but not symmetric. (R and T but not S)*

Example 1.4.11. *On $\mathbb{Z}$ define $a\Re b$, if $a - b$ is even. This relation $\Re$ is an equivalence relation (i.e., it is reflexive symmetric and transitive), but it is not anti-symmetric.*

Example 1.4.12. *On $\mathbb{N}$ define $a\Re b$, if and only if a is a divisor of b. Then the relation $\Re$ is not symmetric but it is reflexive, anti-symmetric and transitive.*

Partial order relation. A relation $\Re$ on a set A is said to be a partial order relation, if $\Re$ is reflexive, anti-symmetric and transitive.

e.g., *set inclusion*: $A \subseteq A$, $A \subseteq B$ and $B \subseteq A \Longrightarrow A = B$ and $A \subseteq B$, $B \subseteq C \Longrightarrow A \subseteq C$ defines a *partial order relation* on the set of all subsets of a given set A.

A set A with a partial order relation $\mathcal{R}$ is called a *partially ordered set* (or a POSET).

The symbol $\leq$ is used to indicate partial order relation; thus a set $\mathcal{A}$ with a partial order relation is written as $(\mathcal{A}, \leq)$.

If the partial order on a set $\mathcal{A}$ be such that for any two elements $a, b \in \mathcal{A}$, either $a \leq b$ or $b \leq a$, then the partial order is said to be a *total order* or a *linear order*.

ORDERED SETS

Q. *What is an Ordered set?*

Let S be a set. An *order* on S is a relation, denoted by $<$, with the following two properties:

(i) If $x \in S$ and $y \in S$, then one and only one of the statements: $x < y$, $x = y$, $y < x$ is true (*Law of Trichotomy*);

(ii) If $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$ (*Law of Transitivity*).

An ordered set is a set S in which an order is defined.

e.g., $\mathbb{Q}$ is an *ordered set,* if $r < s$ is defined to mean that $s - r$ is a positive rational number.

Bounds of an ordered set: *For an ordered set S*, we introduce the concepts of *upper bound* and *lower bound.*

Let $E \subseteq S$. If $\exists\ \beta \in S$ such that $x \leq \beta$ for $\forall x \in E$, then we say that E is *bounded above* and we call β, *an upper bound* of E.

Lower bounds are defined in the same way (with $\geq$ in place of $\leq$).

Definition. Suppose S is an ordered set and $E \subseteq S$, and suppose that E is bounded above. Let $\alpha \in S$ with the following two properties:

(i) α is an upper bound of E;

(ii) If $\gamma < \alpha$, then γ is *not an upper bound* of E.

Then α is called the *least upper bound* (*lub*) of E or the supremum of E and we write $\alpha = \sup E$.

The greatest lower *bound* or infimum of E which is bounded below is defined in a similar manner, i.e., α is a lower bound of E and that no β with $\beta > \alpha$ is a lower bound of E, then $\alpha =$ glb of E or inf E.

These concepts will be used in real analysis.

EXERCISES ON CHAPTER 1–I(A)

(On Basic Concepts)

1. Are the following statements true? Give reasons.

(a) $2 \in \{2, 3\}$;
(b) $3 \in \{4, 1, 5\}$;
(c) $5 \in \{x : x \text{ is a positive integer}\}$;
(d) $\{3, 4\}$ is a subset of $\{3, 4\}$;
(e) $\{a, e, i, o, u\} = \{u, o, i, e, a\}$;
(f) $\{2, 3\} = \{3, 4\}$;
(g) If $a = 3$, and $A = \{x : 3x = 9\}$, then $a = A$.

[*Ans.* (a), (c), (d), (e) are true, others are not true]

2. Let $A = \{1, 2, 3\}$. Check whether the following statements are true or not.

(a) $2 \in A$; (b) $2 \subset A$; (c) $\{2\} \in A$; (d) $\{2\} \subset A$.

[*Ans.* (a) and (d) are true; others are not]

3. Given A is any arbitrary set and ϕ is the null set. State whether the following are true or false.

(a) $\{0\} = \phi$; (b) $\{\phi\} = \phi$; (c) $\{\phi\} = \{0\}$; (d) $\phi \in A$; (e) $\phi \subseteq A$; (f) $A \subseteq A$; (g) $A \in \{A\}$.

[*Ans.* True → (*e*), (*f*) and (*g*); others are false]

4. $A = \{1, 2\}$, $B = \{2, 4, 6\}$, $C = \{1, 3, 4, 6\}$, $D = \{1, 2, 3, 4, 6\}$. Write down a set whose subsets are A, B, C, D.

5. Let $A = \{1, 2, 3\}$, $B = \{1, 2, 4\}$. Obtain the members of the sets: $A \cup B$, $A \cap B$, $A \times B$, $B \times A$.

6. Let $S =$ universal set $= \{1, 2, 3, 4, 5, 6, 7, 8\}$ and A, B, C are its three subsets given by $A = \{1, 5, 6\}$, $B = \{2, 3, 5, 7, 8\}$ and $C = \{1, 3, 6, 8\}$. Obtain the following sets: $A \cap B' \cap C$ and $(A \cup C) \cap (B \cup C')$.

7. Let $A = \{1, 2, 3, \{4, 5, 6\}\}$ and $B = \{1, 2, \{4, 6\}\}$. Find $A \cap B$ and $A \cup B$.

8. Let $A = \{1, 2, 3\}$ and $B = \{1, 5\}$. Obtain the Cartesian products $A \times B$, $B \times A$, $A \times A$, $B \times B$, $(A \times B) \times A$ and $A \times (B \times A)$.

9. Define equality of two sets. For n sets let $A_1, A_2, \cdots, A_n$. Let $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots \subseteq A_n$ and also let $A_1 \supseteq A_n$. Prove that the n sets are all equal.

10. Let $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 4, 6, 8, 10\}$ and $C = \{3, 4, 5, 6\}$.
Obtain the sets: $(A - B)$; $(B - C)$; $(C - A)$; $(B - A)$; $(B - B)$.

11. Let $A = \{1, 1/2, 1/3, 1/4\}$, $B = \{1/2, 1/4, 1/6, 1/8\}$, $C = \{1/3, 1/4, 1/5, 1/6\}$ and suppose that the universe is $S = \{1, 1/2, 1/3, 1/4, \cdots, 1/9\}$.
Obtain: $(A \cap C)'$, $(A \cup B)'$, $(A')'$, $(B - C)'$, $B - A$, $B' - A'$, $A' \cap B$, $A \cup B'$, $A' \cap B'$.

(On applications of the Laws of Algebra of Sets)

1. A, B, C are any three sets. Verify the following properties:

(a) $A - B = A - (A \cap B) = (A \cup B) - B$;

(b) $(A - B) - C = A - (B \cup C)$;
(c) $A - (B - C) = (A - B) \cup (A \cap C)$;
(d) $(A \cup B) - C = (A - C) \cup (B - C)$;
(e) $A - (B \cup C) = (A - B) \cap (A - C)$;
(f) $A \cap (B - C) = (A \cap B) - (A \cap C)$;
(g) $(A - C) \cap (B - C) = (A \cap B) - C$.

Remember: $A - B = A \cap B'$; use this fact whenever necessary.

2. The *symmetric difference* of two sets A and B (denoted by $A \Delta B$) is given by $A \Delta B = (A - B) \cup (B - A)$. Verify the following properties:
(a) $A \Delta B = (A \cup B) - (A \cap B)$;
(b) $A \Delta (B \Delta C) = (A \Delta B) \Delta C$; (Associativity)
(c) $A \Delta \Phi = A$; $A \Delta A = \Phi$;
(d) $A \Delta B = B \Delta A$; (Commutativity)
(e) $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$. (Distributive Law)

3. Prove by using laws of Algebra of sets:
(a) $(A \cap B) \cup (A \cap B') = A$;
(b) $(A \cap B') \cup B = A \cup B$;
(c) $A \cap \Phi = \Phi$; $(A \cap B') \cap A' = \Phi$;
(d) $(A' \cup C) \cap (B' \cup D') = (B' \cup A') \cup (B' \cap C) \cup (D' \cap A') \cup (D' \cap C)$;
(e) $A \cup (A \cap B) = A$; $A \cup (A' \cap B) = A \cup B$;
(f) $A \cap (A' \cup B) = A \cap B$; $A \cap (A \cup B) = A$;
(g) $(A \cup B) \cap (B \cup C) \cap (C \cup A) = (A \cap B) \cup (A \cap C) \cup (B \cap C)$.

(On Relations on Sets)

1. Define a relation $\Re$ on the set P of all people by taking $x \Re y$ to mean x and y have the same age.
Is it an equivalence relation? Justify your assertion.

2. Show that the relation $>$ (greater than) on the set of real numbers is *Transitive* but neither *Reflexive* nor *Symmetric*.

3. Give an example of a relation on a set:
(a) which is symmetric and transitive but not reflexive.
(b) which is reflexive and symmetric but not transitive.
(c) which is symmetric but neither reflexive nor transitive.

[*Ans.* (a) $x \Re y$, only when $x - y \neq 0$ $(x, y \in \mathbb{R})$ | $\mathbb{R}$ is the set of all real numbers. $\mathbb{Z}$ is the set of all integers]
Another example: Ex. 1.4.9
(b) Ex. 1.4.3.
(c) Ex. 1.4.7.]

4. Let $\mathbb{Z}$ be the set of all integers. We define a relation "$\equiv$" (called *Congruence relation*): If $a, b \in \mathbb{Z}$, then $a \equiv b$ (mod 5), iff $a - b$ is divisible by 5. Prove that '$\equiv$' is an equivalence relation on $\mathbb{Z}$.

5. Find which of the following relations are reflexive, symmetric, transitive or equivalence relation: [$\mathbb{R}, \mathbb{S}, \mathbb{T}$ or $\mathbb{EQ}$]

Set	*Relation*
I. Set of all triangles in a plane	(a) is congruent to (b) is similar to
II. Set of all lines in a plane	(a) is perpendicular to (b) is parallel to
III. Set of all integers	(a) $a \Re b$, iff $\lvert a - b \rvert \leq b$ (b) $a \Re b$, iff $3a + 4b$ is divisible by 7 (c) $a \Re b$, iff $a - b$ divisible by 5

[*Ans.* I. (a) $\mathbb{EQ}$; (b) $\mathbb{EQ}$. II. (a) $\mathbb{S}$ (not $\mathbb{R}, \mathbb{T}$); (b) $\mathbb{EQ}$. III. (a) Not $\mathbb{R}, \mathbb{S}, \mathbb{T}$; (b) $\mathbb{EQ}$; (c) $\mathbb{EQ}$.]

1.5 Mapping or Function

We now discuss the most fundamental notion of analysis, namely *function* (or *mapping*).

A *function* f from a set A into a set B is a *rule of correspondence* that assigns to each element $x \in A$, a uniquely determined element $f(x)$ in B. We also call it a *mapping* from A into B and write $f : A \to B$. (*read*: f maps A into B).

In the definition of function given above we have used a phrase *rule of correspondence* which needs further clarification. So the following definition of function is more widely accepted:

Definition. Let A and B be two non-empty sets. Then *a function* (or *a mapping*) from A to B is a set f of ordered pairs in $A \times B$ such that for each $x \in A$ there exists a unique $y \in B$ with $(x, y) \in f$. (This means that if $(x, y) \in f$ and $(x, y') \in f$, then $y = y'$). A function f from A to B is a relation from A to B such that no two elements of f have the same first component.

The set A of the first elements of a function is called the *domain* of f and the set of all the second elements of f is called the *range* of f. Note that dom $f = A$ but range of $f \subseteq B$. Range of f is also denoted by $f(A)$.

Example 1.5.1. *Let $\mathbb{R}$ be the set of all real numbers. Suppose f maps $\mathbb{R}$ into $\mathbb{R}$ (i.e., $f : \mathbb{R} \to \mathbb{R}$) defined by $f(x) = x^2$, $x \in \mathbb{R}$. What are the values of the function at $x = 0, -1, 2, -3$?*

Clearly, $f(0) = 0$, $f(-1) = (-1)^2 = 1$, $f(2) = 4$, $f(-3) = 9$. The domain of f is $\mathbb{R}$ and the co-domain is also $\mathbb{R}$. Notice that $x^2 \geq 0\ \forall x \in \mathbb{R}$. So the range of f is $\mathbb{R}^+$, the set of all positive real numbers and the singleton set $\{0\}$, i.e., $f(\mathbb{R}) = \mathbb{R}^+ \cup \{0\}$.

1.6 Different types of Mappings

I. ONTO Mapping (or Surjective Mapping). The map $f : A \to B$ is called *surjective* (or ONTO) *iff* $f(A) = B$, i.e., range of f = co-domain of f, i.e., $\forall y \in B, \exists\ x \in A$ such that $f(x) = y$.

II. One-One mapping (or Injective mapping). The map $f : A \to B$ is called *one-one mapping* (or *injective mapping*), if and only if distinct members of A have distinct images in B, i.e., for $x_1, x_2 \in A$, $f(x_1) = f(x_2) \Longrightarrow x_1 = x_2$ or the contra positive statement: $x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2)$.

III. Bijective mapping. The map $f : A \to B$ is said to be a *bijective mapping* or a *one-one* onto mapping or *one-one correspondence* if it is both *injective* and *surjective*, i.e., if every $x \in A$ has a unique image $y \in B$ and every $y \in B$ has a unique pre-image $x \in A$. A bijective mapping is also called a *bijection*.

Example 1.6.1. *Let* $f : \mathbb{R} \to \mathbb{R}$, *defined by* $f(x) = x^2$, $\forall x \in \mathbb{R}$.

- This mapping is *not injective* (see that $f(1) = 1$ and $f(-1) = 1$; again $f(2) = 4$ and $f(-2) = 4$, i.e., it is not true that distinct members of the domain have distinct images).
- This mapping is not *surjective* (see that $\exists\ -1 \in$ co-domain $\mathbb{R}$ which has no pre-image x in the domain $\mathbb{R}$ because every image is non-positive).

Thus the mapping cannot be a *bijective mapping*.

Example 1.6.2. *Let* $f : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$, *defined by* $f(x) = x^2$, $x \in \mathbb{R}^+ \cup \{0\}$. *Verify that this mapping is injective but not surjective.*

Example 1.6.3. *Let* $f : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$, *defined by* $f(x) = x^2$, $\forall x \in \mathbb{R}$. *Check: This mapping is not injective but it is surjective.*

Example 1.6.4. *Let* $f : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ *defined by* $f(x) = x^2$, $\forall x \in \mathbb{R}^+ \cup \{0\}$. *Verify that this mapping is both injective and surjective, i.e., it is a bijective mapping.*

Remember: (i) In order to prove that the mapping $f : A \to B$ is an injective mapping, we must establish that $\forall x_1, x_2 \in A$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$. To do this, start with $f(x_1) = f(x_2)$ and show that $x_1 = x_2$. f is injective if each element of B has at most one pre-image.

(ii) To prove $f : A \to B$ is a surjective mapping, we must show that for any $y \in B$, $\exists$ at least one $x \in A$ such that $f(x) = y$. f is surjective if each element of B has at least one pre-image.

(iii) To prove $f : A \to B$ is a bijection, we are to show that each element $y \in B$ has exactly one pre-image $x \in A$.

Example 1.6.5. *Let* $A = \{x : x \in \mathbb{R}$ *but* $x \neq 1\}$. *Define* $f(x) = 2x/(x-1)$, $x \in A$.

Solution: To prove that the function f is injective, we start with $f(x_1) = f(x_2)$, where $x_1, x_2 \in A$, i.e., $2x_1/(x_1-1) = 2x_2/(x_2-1) \Longrightarrow x_1(x_2 - 1) = x_2(x_1 - 1) \Longrightarrow x_1 = x_2$.

$\therefore f$ is injective.

To determine the range of f we solve for x the equation $\frac{2x}{(x-1)} = y$. We obtain $x = \frac{y}{(y-2)}$ which is defined for $y \neq 2$.

Thus range of $f = \{y : y \in \mathbb{R}$ and $y \neq 2\} = B$ (say). Then $f : A \to B$ is a bijection.

Example 1.6.6. *Let* $f : \mathbb{Z} \to \mathbb{Z}$ *defined by* $f(x) = x + 2$, $x \in \mathbb{Z}$. *See that this mapping is both injective and surjective, i.e., f is bijective.*

IV. Inverse mappings (or Inverse functions)

Definition. If $f : A \to B$ is a bijection of A ONTO B, then the set

$$g = \{(b, a) \in B \times A : (a, b) \in A \times B\}$$

is a function on B into A. This function is called the *inverse function* (or *inverse mapping*) of f, and is denoted by f^{-1}. The function f^{-1} is also called the *inverse* of f.

Note: In order to define inverse of $f : A \to B$, f must be bijective and domain (f) = range of f^{-1} and range of f = domain of f^{-1}. Also $y = f(x)$, if and only if $x = f^{-1}(y)$.

Example 1.6.7. *We have observed in Example* 1.6.4, *that the function*

$$f : R^+ \cup \{0\} \to R^+ \cup \{0\}$$

defined by $f(x) = x^2, x \in R^+ \cup \{0\}$ *is both injective and surjective* (*i.e., here f is a bijection on* $R^+ \cup \{0\}$) *and hence* f^{-1} *exists.*

The function inverse to f is given by (solving $y = x^2$ give $x = \sqrt{y}$)

$$f^{-1}(y) = \sqrt{y} \text{ for } y \in R^+ \cup \{0\} \text{ (range of } f).$$

We may write $f^{-1}(x) = \sqrt{x}$ for $x \in R^+ \cup \{0\}$ (replacing y by x).

Example 1.6.8. *See Example* 1.6.5. $f : A = \{x \in \mathbb{R} : x \neq 1\} \to B = \{y \in \mathbb{R} : y \neq 2\}$ *defined by* $f(x) = 2x/(x-1)$, $x \in A$.

This function f is a bijection of A ONTO B. Hence f^{-1} exists and f^{-1} is given by $f^{-1}(y) = y/(y-2)$, $y \in B$ [Solving $y = 2x/(x-1)$ gives $x = y/(y-2)$].

We may also write $f^{-1}(x) = x/(x-2)$, $x \in B$ (replacing y by x).

V. Identity mapping (or Identity function). The function $f : A \to A$ defined by $f(x) = x$, $x \in A$ is called the *identity function* on A, denoted by I_A. Then I_A keeps every element of A fixed.

VI. Equality of two mappings. Let $f : A \to B$ and $g : A \to B$ be two functions, both defined on A. Then f and g are called *equal* (written as $f = g$), if $f(x) = g(x)$ $\forall x \in A$.

Note: For equality, f and g must have the same domain and for each $x \in A$, $f(x) = g(x)$.

Example 1.6.9. *Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$, $\forall x \in \mathbb{R}$ and let $g : \mathbb{R} \to \mathbb{R}$ be defined by*

$$g(x) = \begin{cases} x, & \text{when } x \geq 0 \\ -x, & \text{when } x < 0. \end{cases}$$

Both f and g have the same domain and moreover, $f(x) = g(x)$ $\forall x \in \mathbb{R}$.

$\therefore$ *in this case, we write $f = g$.*

VII. Composition of function (or Composite mappings). To compose two functions f and g, we first find $f(x)$ and then apply g-rule to $f(x)$ and obtain $g\{f(x)\}$. Obviously this is not possible unless $f(x)$ is an element in the domain of g. In order to be able to do this *for all* $f(x)$ we are to assume that the range of f is contained in the domain of g, i.e., range of $f \subseteq$ domain of g.

Definition. Let $f : A \to B$ and $g : C \to D$ be two mappings. If the range $f(A)$ of f is a subset of C then the mapping $g \circ f : A \to D$ defined by $(g \circ f)(x) = g(f(x)), x \in A$ is called a *composite mapping.*

In particular if $f : A \to B$ and $g : B \to C$ then the composite mapping $g \circ f$ is always possible as $f(A) \subseteq B$. The order of the compositions should be maintained strictly, because $g \circ f$ and $f \circ g$ are different functions, in general, when both are defined.

Example 1.6.10. *Let f and g be two functions defined on $\mathbb{R}$, given by $f(x) = 3x$ and $g(x) = 2x^2 - 1$.*

Since dom $g = \mathbb{R}$ and range of $f \subseteq$ dom $g = \mathbb{R}$, the domain of $g \circ f$ is also equal to $\mathbb{R}$ and the composite function $g \circ f$ is defined by

$$(g \circ f)(x) = g\{f(x)\} = g(3x) = 2(3x)^2 - 1 = 18x^2 - 1.$$

On the other hand, $(f \circ g)x = f\{g(x)\} = f(2x^2 - 1) = 3(2x^2 - 1) = 6x^2 - 3.$

Thus we observe $g \circ f \neq f \circ g$.

Most important point to construct $g \circ f$ is to see that the range of f is contained in the domain of g.

e.g., let $f(x) = 1 - x^2$ and $g(x) = \sqrt{x}$, domain of $g = \{x : x \in \mathbb{R}, x \geq 0\}$.

The composite function $g \circ f$ is given by $(g \circ f)x = g\{f(x)\} = g(1 - x^2) = \sqrt{1 - x^2}$. Only for $x \in$ domain of f that satisfies $f(x) \geq 0$, i.e., for those x which satisfies $-1 \leq x \leq 1$.

VIII. Restrictions of functions. Suppose f maps A into B and let $A_1 \subset A$. We define $f_1 : A_1 \to B$ by the rule $f_1(x) = f(x)$ $\forall x \in A_1$. This function f_1 is called the *restriction of f to A_1*, denoted by f_{A_1}. We thus *cut down in size* the domain of a function. Of course, there are good reasons for restricting the domain in this manner.

A very common example: Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$, for $x \in \mathbb{R}$. This function is certainly not injective $\{\because\ f(-1) = 1, f(1) = 1\}$ and hence not bijection and so it cannot have an inverse.

However, if we restrict f to set $A_1 = \{x : x \in \mathbb{R}, x \geq 0\}$ or $A_1 = \mathbb{R}^+ \cup \{0\}$, then the restriction function f_{A_1} is both injective and surjective so that the restriction function f_{A_1} has an *inverse function* (*positive square root function*).

The trigonometric functions $\sin x$ and $\cos x$ are not injective for all $x \in \mathbb{R}$. However, by making suitable restrictions of these functions, one can obtain the *inverse sine* and *inverse cosine* functions. Sine-function can be restricted to $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ and cosine function can be restricted to $0 \leq x \leq \pi$. In these restrictive domains $\sin^{-1} x$ and $\cos^{-1} x$ are defined.

Note: The function $f : A \to B$ is an extension of its restriction

$$f_1 : A_1 \to B \text{ then } A_1 \subset A$$

EXERCISES ON CHAPTER 1: I(B)

(On Mappings)

(Hints are given at the end of this Exercise for *-marked problems)

1. Define mapping of a set X into a set Y. Do the following correspondences conform to your definition? If so, mention the Range or Image set.
   (a) $f : \mathbb{Z}^+ \to E$, defined by $f(x) = 2x$, $\forall x \in \mathbb{Z}^+$. ($\mathbb{Z}^+$ **is the set of all positive** integers and E is the set of even positive integers.)
   (b) $g : \mathbb{R} \to \mathbb{R}$, defined by $g(x) = e^x$, $\forall x \in \mathbb{R}$.
   (c) $f : X \to Y$ (X = set of all students of your college and Y = set of ages of the students in years).
   (d) $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = \log x$, $x \in \mathbb{R}$.